INDIVIDUAL DIFFERENCES IN THOUGHT PROCESSES AND THEIR RELEVANCE TO TEACHING MATHEMATICS

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Individual Differences in Thought Processes and Their Relevance to Teaching Mathematics

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For years mathematics education journals have been filled with articles on the usefulness of visual aids; so many teachers have found visual techniques helpful that their effectiveness is virtually folk wisdom now. However, a rapidly increasing body of research on individual differences in thought processes calls into question this long-held belief. There is evidence that while some students think and reason visually, through forming images, for others the avenue to the most effective learning is through the auditory and kinesthetic senses. This suggests that, contrary to folk wisdom, teaching would be improved if students who reason non-visually received instruction of a verbal-auditory type rather than using a visual approach with all students. This article will present some of the research on this topic and illustrate how the different modes of thinking can be applied in teaching some topics in secondary mathematics.

Early studies in memory and learning showed differences between "visualizers" and "non-visualizers" (Bartlett, 1932; Mulholland, 1964; Stuart, 1963). Evidence on the proportion of people in these categories points to about a normal curve distribution of individuals along the spectrum of visual and non-visual orientations (Lowenfeld, 1939; Fernald, 1943; Day and Wedell, 1972).

An important consideration for educators is the degree to which students learn more effectively when lesson presentations are in their "preferred" mode of thought (a student's "preferred" modality is the one in which he has a higher aptitude). Some studies have failed to find aptitude-treatment interaction effects significant (Cripe, 1966; Davis, 1967; Becker, 1967). Perhaps these experiments yielded negative results because the distinction between instructional treatments (i.e., verbal and visual instruction materials) was not sharp enough or the measures of visual and non-visual aptitudes were not valid.

However positive evidence of aptitude-treatment interaction has been found in a number of experiments. Behr (1967) studied 228 college-age students in an elementary mathematics course and found a correlation between semantic aptitude and ability to learn modulus seven arithmetic best from a verbal instructional unit. There was also a tendency for figural (visual) ability to follow success in learning from diagrammatic instruction. Kropp, Nelson and King (1967) found that in their sample of 400 elementary school students, those with more deductive than inductive reasoning ability learned better from a deductive presentation of basic set concepts. Likewise those with higher inductive ability learned better from the inductive presentation ($p < .05$). No verbal-figural interaction was found, however. Carry (1968) studied 191 high school geometry students, each of whom learned about quadratic inequalities from a graphical or analytical instructional unit. Following instruction, students were given a learning and a transfer test. The former did not support the hypothesis of aptitude-treatment interaction, but the latter did ($p < .01$).

Another important question for teachers is the effect on all students (regardless of their preferred modality) of having both the lesson presentation and the test be visual or both verbal versus mixing the two. Duncan and Hartley (1969) found that subjects who were tested in the same form in which they had received instruction did significantly better ($p < .001$) than subjects who had to translate the information into the other form. A similar study by Jenkins *et al.* (1967) made the same finding. This has clear implications for the relationship between methods of teaching and testing.

Some researchers have tried to develop an individual's ability to think in the modality in which he is deficient. But these efforts have been largely fruitless (Fernald, 1943 and Goins, 1958). This suggests that thinking modality may be genetic or developed at an early age.

Indeed there is evidence that the brain waves of visual-and non-visual-thinking individuals differ. One of the earliest findings of the EEG was that the alpha brain rhythms are prominent in most people when their eyes are shut and their mind is at rest. Alpha rhythms usually disappear when the eyes are open or the individual is making some mental effort, such as calculating an addition problem. But people who tend to use auditory, kinesthetic, or tactile perceptions are those whose alpha rhythms continue even when their eyes are open and their mind is active. On the other hand, people who think visually have few alpha rhythms even when their eyes are shut and their mind is idle. Finally, in the intermediate group are people without a strong preference for either visual or non-visual thinking patterns.

Walter studied several groups of over six hundred people and concluded that about one-sixth of the population are visualizers, one-sixth are non-visualizers and two thirds are in the intermediate group. According to Walter, the following test indicates to which group one belongs.

> Shut your eyes. Think of a wooden cube like a child's block. It is painted. Now imagine that you cut this in halves across one side, then cut these halves in halves, and then cut them a third time at right angles. Now, think of the little cubes you have made. How many of their sides will be unpainted?
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> Did you work it out or did you "see it"? Then what else did you see? What color was the cube? Did you see sawdust falling as you cut it (Walter, 1953, p. 216)?

Walter claims that in solving the problem, a visualizer would have pictured more details than are necessary, while a non-visualizer would see no picture at all. A person in the intermediate group would see a picture that is just clear enough to answer the question.

Walter reports that the performance of the visualizers is "rapid and precise" if they are given a problem that can be solved through visualization. But if they are given an abstract problem or one that requires too elaborate a mental picture, then they become "sluggish and confused." The non-visualizers do not use visual images in their thinking unless it is necessary to. And even then, Walter claims, "their mind's eye is almost always blind; they think in abstract terms or in sounds. . . . " (Walter, 1953, p. 217). People in the intermediate group do not always use visual images to think things through, but they can conjur up a picture when it is expedient.

It behooves us as teachers to consider the relevance of these ways of thinking to mathematics education. Many topics in high school mathematics lend themselves to both visual and non-visual interpretations. Consider the solution of $x + 10 \; = 4$. One can conceive of this purely in algebraic symbolization in which case the student may be taught to solve the equation as follows. By definition of absolute value, $x + 10 \; = 4$ implies that $^{+}(x + 10) = 4$ or that $^{-}(x + 10) = 4$. Then these equations can be solved by the usual algebraic manipulation.

$^{+}(x + 10) = 4 \Rightarrow$	or	$^{-}(x + 10) = 4 \Rightarrow$
$x = 4 - 10 \Rightarrow$		$^{-}x - 10 = 4 \Rightarrow$
$x = -6$		$^{-}x = 4 + 10 \Rightarrow$
		$(^{-}1)(^{-}x) = (14)(^{-}1) \Rightarrow$
		$^{+}x = {}^{-}14.$

On the other hand, the meaning of the equation $x + 10 \; = 4$ can be pictured. Clearly $x + 10 \; = 4$ is the same thing as $x - {}^{-}10 \; = 4$, therefore the distance between x and $^{-}10$ is 4 in either direction. Picturing or drawing a number line at this point enables the student to "see" the solutions.

Another example is the process of interpolation which high school students do with both logarithms and trigonometric functions. Algebra books generally present this in the algebraic mode by developing a formula for finding the solution. For example, suppose in solving for an angle a student finds that the cosine of that angle is .9935. He finds from a table of trigonometric ratios that the cosine of 6° is .9945 and the cosine of 7° is .9925.

Use of an algebraic formula is a methodical way of solving the interpolation. One subtracts .9935 from the cosine of 6° and divides that by the difference between the cosine of 6° and the cosine of 7°. Then the ratio is multiplied by 60 minutes and the result is added to 6°. Hence,

$$6^\circ + \frac{(.9945 - .9935)}{(.9945 - .9925)} \cdot (60 \text{ minutes}) =$$

$$6^\circ + \frac{(.0010)}{(.0020)} \cdot (60 \text{ minutes}) =$$

$$6^\circ + (\tfrac{1}{2}) \cdot (60 \text{ minutes}) =$$

$$6^\circ + 30 \text{ minutes} =$$

$$\text{So } \Theta = 6^\circ \, 30'.$$

On the other hand, in this problem one could point out that the value obtained is exactly half-way between the tabled values hence by linear interpolation the solution would be half-way between the angles. Of course not all problems are so simple. But the student can learn to think of the difference between the two tabled values as a unit and think what fraction of the "distance" between them the obtained value falls at.

For example in the same problem suppose the student found the cosine of Θ to be .9930. Since the tabled values are 20 units apart, the obtained value is $15/20 = 3/4$ of the way from the cosine of 6° to the cosine of 7°. Hence ¾ of an angle must be added to 6°. Likewise one could say that the obtained value is ¼ of the way from the cosine of 7° to the cosine of 6° and so ¼ of an angle must be subtracted from 7°. A student who pictures this process avoids the common problem of remembering which angle to add or subtract from to get the final answer.

This method may be difficult though, if the numbers do not readily reduce to a simple fraction. But if a student works a few easy problems first, he "sees" what is going on and can then work problems with awkward numbers.

Interpolation problems are easy to interpret algebraically in terms of symbols, or geometrically by visualizing. Yet when interpolation is taught in algebra or algebra-trig courses, there is a mental set to use the algebraic approach to the exclusion of the geometric one. Although not all topics in high school mathematics lend themselves as nicely as these to both kinds of interpretations, still in many areas of mathematics the two ways of thinking do apply to a reasonable extent. Since some people think in one vein and some in another, it is important for mathematics teachers to cover topics from both angles (especially if the text takes only one approach).

Particularly with slow learners, mathematics teachers often encourage visualization. Yet this may not be the best approach for a given student. In fact, in order for each student to get as much input as possible in his accustomed vein of thought, it may be warranted to match teachers with students who share their thinking patterns. This is an area of consideration which has potential for refining teaching methods to make them of greatest benefit to each student.

BIBLIOGRAPHY

1. Bartlett, F. C. *Remembering.* London: Cambridge Univ., 1932.
2. Becker, Jerry P. *An Attempt to Design Instructional Techniques in Mathematics to Accommodate Different Patterns of Mental Ability.* Doctoral thesis, Palo Alto: Stanford Univ., 1967, Abstract: *Dissertation Abstracts,* 28:957; No. 3A, 1967.
3. Behr, Merlyn J. *A Study of Interactions Between "Structure of Intellect" Factors and Two Methods of Presenting Concepts of Modulus Seven Arithmetic.* Doctoral thesis, Tallahassee: Florida State Univ., 1967, Abstract: *Dissertation Abstracts,* 28:1693-A, 1967.
4. Carry, Laroy Ray. *Interaction of Visualization and General Reasoning Abilities with Instructional Treatment in Algebra.* Doctoral thesis, Stanford, Calif.: Stanford Univ., 1968, Abstract: *Dissertation Abstracts,* 29: 475A; No. 7, 1968.
5. Cripe, Antje G. *Auditory and Visual Learning Related to Illinois Test of Psycholinguistic Abilities Sensory Channels.* Doctoral Dissertation, Lafayette: Purdue Univ., 1966.
6. Davis, John. *An Investigation of the Interactions of Certain Instructional Strategies with the Structure of Basic Mental Abilities in the Learning of Some Mathematical Operations.* Doctoral thesis, Tallahassee: Fla. State Univ., 1967.
7. Day, J. B. and Wedell, D. "Visual and Auditory Memory in Spelling: An Exploratory Study." *The British Journal of Educational Psychology,* 42, part 1, Feb. 1972, 33-9.
8. Duncan, Charlotte and Hartley, James. "The Effect of Mode of Presentation and Recall on a Simple Learning Task." *Programmed Leaning and Educational Technology,* 6, 1969, 154-8.
9. Fernald, Grace M. *Remedial Techniques in Basic School Subjects.* New York: McGraw-Hill Book Co., 1943.
10. Goins, Jean. *Visual Perceptual Abilities and Early Reading Progress.* University of Chicago Press, Supplementary Educational Monographs, #37, Feb. 1958.
11. Huebner, Robert. *Interactions Between Patterns of Individual Differences in Sensory Modalities of Students and Methods of Classroom Instruction.* Doctoral theses, College Park: University of Maryland, 1969.
12. Jenkins, J. R.; Neale, D. C. and Deno, S. L. "Differential Memory for Picture and Word Stimuli." *Journal of Educational Psychology,* 58, 303-7.
13. Kropp, R. P.; Nelson, W. H. and King, F. J. *Identification and Definition of Subject-Matter Content Variables Related to Human Aptitudes,* Cooperative Research Project No. 2117. Tallahassee: Fla. State Univ., 1967.
14. Lowenfeld, Viktor. *The Nature of Creative Activity.* New York: Harcourt Brace, 1939.
15. Mulholland, Ann M. "The Impact of Individual Differences on Language Learning." *Exceptional Children,* April 1967, p. 359-63.
16. Stafford, Richard. "Negative Relationships Between Ability to Visualize Space and Grades in Specific Courses." *Journal of Learning Disabilities,* 5, no. 1, Jan. 1972, 38-40.
17. Stuart, Marion Fenwick. *Neurophysiological Insights Into Teaching.* Palo Alto, Cal.: Pacific Books, 1963.
18. Vaughn, K. W. "The Yale Scholastic Aptitude Tests as Predictors of Success in the College of Engineering." *Journal of Engineering Education,* 34, 1943, 572-82.
19. Walter, W. G. *The Living Brain.* New York: W. W. Norton and Co., 1953.